2023-24 MATH2048: Honours Linear Algebra II

Homework 8

Due: 2023-11-13 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V be an inner product space over F, show that

   (a) If $x, y \in V$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

   (b) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$ (The *parallelogram law*).

   (c) Let $v_1, v_2, \ldots, v_k$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k \in F$. Then $\| \sum_{i=1}^{k} a_i v_i \|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2$.

2. Prove that if V is an inner product space, then $|\langle x, y\rangle = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other. Try to derive a similar result for the equality $\|x+y\| = \|x\| + \|y\|$.

3. Let $V = M_{2\times 2}(\mathbb{C})$. Apply the Gram–Schmidt process to

$$S = \left\{ \begin{pmatrix} 1-i & -2-3i \\ 2+2i & 4+i \end{pmatrix}, \begin{pmatrix} 8i & 4 \\ -3-3i & -4+4i \end{pmatrix}, \begin{pmatrix} -25-38i & -2-13i \\ 12-78i & -7+24i \end{pmatrix} \right\}$$

   to obtain an orthogonal basis S' for *span*(S). Then normalize the vectors in S' to obtain an orthonormal basis S''.

4. Let V be a finite-dimensional inner product space over F.

   (a) *Parseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$ prove that $\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$.

   (b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot \rangle$, then for any $x, y \in V$, we have $\langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle$, where $\langle \cdot, \cdot\rangle'$ is the standard inner product on F^n.

5. (a) *Bessel's Inequality.* Let V be an inner product space, and let $S = v_1, v_2, \ldots, v_n$ be an orthonormal subset of V. Prove that for any $x \in V$ we have $\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$.

   (b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in span(S)$.

The following are extra recommended exercises not included in homework.

1. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i. Hint: Use mathematical induction on k.

2. Let T be a linear operator on a vector space V, and let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V. Prove that $W_1 + W_2 + \cdots + W_k$ is also a T-invariant subspace of V.

3. Let T be a lineare operator on a finite-dim vector space V, and let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Prove that

$$\det(T) = \det(T_{W_1}) \det(T_{W_2}) \cdots \det(T_{W_k})$$

4. Provide reasons why each of the following is not an inner product on the given vector spaces.

   (a) $\langle (a, b), (c, d) \rangle = ac - bd$ on $\mathbb{R}^2$.

   (b) $\langle A, B \rangle = tr(A + B)$ on $M_{2\times 2}(\mathbb{R})$.

   (c) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t)dt$ on $P(\mathbb{R})$.

5. Let β be a basis for a finite-dimensional inner product space.

   (a) Prove that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$.

   (b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

6. Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

7. Let V be an inner product space over F. Prove the *polar identities*: For all $x, y \in V$,

   (a) $\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = \mathbb{R}$.

   (b) $\langle x, y \rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$ if $F = \mathbb{C}$, where $i^2 = -1$.

8. Let $V = F^n$ and let $A \in M_{n\times n}(F)$.

(a) Prove that $\langle x, Ay\rangle = \langle A^* x, y\rangle$ for all $x, y \in V$.

(b) Suppose that for some $B \in M_{n\times n}(F)$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $T(x) = Ax$ and $U(x) = A^* x$. Show that $[U]_\beta = [T]_\beta^*$ for any orthonormal basis β for V.